LINEAR ALGEBRA CONTINUED

$$A = \begin{pmatrix} 2 & 0 & -1 & 2 \\ 6 & 5 & 3 & 7 \\ 2 & -1 & 0 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & \sim \\ 0 & 1 & 0 & \sim \\ 0 & 0 & 1 & \sim \end{pmatrix}$$

- Process to solve is called row reduction.

a) Use original set of equations → modify to eliminate x in equations 2 and 3

$$\begin{aligned} 2x \qquad\quad - z &= 2 \\ 5y + 6z &= 1 \\ -y + z &= 2 \end{aligned} \qquad \begin{pmatrix} 2 & 0 & -1 & 2 \\ 0 & 5 & 6 & 1 \\ 0 & -1 & 1 & 2 \end{pmatrix}$$

$$-3(2x \qquad - z = 2)$$

$$\begin{aligned} \cancel{6x} + 5y + 3z &= 7 \\ + \; -\cancel{6x} + 0y + 3z &= -6 \\ \hline \boxed{5y + 6z = 1} \end{aligned}$$

b) Convenient to interchange second and third equations:

$$\begin{aligned} 2x \quad\quad - z &= 2 \\ -y + z &= 2 \\ 5y + 6z &= 1 \end{aligned} \qquad \begin{pmatrix} 2 & 0 & -1 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & 5 & 6 & 1 \end{pmatrix}$$

c) Use second equation to eliminate y terms from other equations

$$\begin{aligned} 2x \quad\quad - z &= 2 \\ -y + z &= 2 \\ +11z &= 11 \end{aligned} \qquad \begin{pmatrix} 2 & 0 & -1 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 11 & 11 \end{pmatrix}$$

d) Divide third equation by 11 and use it to eliminate z-terms from other equations.

$$\begin{aligned} 2x &= 3 \\ -y &= 1 \\ z &= 1 \end{aligned} \qquad \begin{pmatrix} 2 & 0 & 0 & 3 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

e) Customary to divide by leading coefficient

$$\begin{pmatrix} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad \begin{aligned} x &= 3/2 \\ y &= -1 \\ z &= 1 \end{aligned}$$

- Elementary Row Operations:
  - Interchange two rows
  - Multiply or divide by a (nonzero) constant
  - Add a multiple of one row to another

$$\begin{pmatrix} 1 & 0 & 0 & \sim \\ 0 & 1 & 0 & \sim \\ 0 & 0 & 1 & \sim \end{pmatrix}$$

Augmented Matrix

Example 2

$$\begin{aligned} x - y + 4z &= 5 \\ 2x - 3y + 8z &= 4 \\ x - 2y + 4z &= 9 \end{aligned}$$

$$\begin{pmatrix} 1 & -1 & 4 & 5 \\ 2 & -3 & 8 & 4 \\ 1 & -2 & 4 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 4 & 5 \\ 0 & -1 & 0 & -6 \\ 0 & -1 & 0 & 4 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 4 & 11 \\ 0 & -1 & 0 & -6 \\ 0 & 0 & 0 & -20 \end{pmatrix}$$

$\varnothing$ $z = 20$!!!

These equations are inconstent!

- Rank of a Matrix
  - Number of nonzero rows when a matrix has been row reduced
  - Example (from Example 2)

$$A = \begin{pmatrix} 1 & 0 & 4 & 11 \\ 0 & -1 & 0 & -6 \\ 0 & 0 & 0 & -20 \end{pmatrix} \quad \text{Rank} = 3$$

(all parts)

$$M = \begin{pmatrix} 1 & 6 & 4 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{Rank} = 2$$

(coefficients)

  - If Rank $M <$ Rank A, equations are inconsistent and there is no solution.
  - If Rank $M =$ Rank A, there is one solution

Solve:

$$\begin{aligned} 2x + 5y + z &= 2 \\ x + y + 2z &= 1 \\ x \quad\;\; + 5z &= 3 \end{aligned}$$

Solution:

$$\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad \begin{aligned} x &= -2 \\ y &= 1 \\ z &= 1 \end{aligned}$$

$$\begin{pmatrix} 2 & 5 & 1 & 2 \\ 1 & 1 & 2 & 1 \\ \overset{-2}{\cancel{1}} & 0 & \overset{-10}{\cancel{5}} & \overset{-6}{\cancel{3}} \end{pmatrix}$$

$$\begin{pmatrix} 2 & 5 & 1 & 2 \\ \overset{-2}{\cancel{1}} & \overset{-2}{\cancel{1}} & \overset{-4}{\cancel{2}} & \overset{-2}{\cancel{1}} \\ 0 & 5 & -9 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 5 & 1 & 2 \\ 0 & \overset{-5}{\cancel{3}} & \overset{5}{\cancel{3}} & 0 \\ 0 & 5 & -9 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 5 & 1 & 2 \\ 0 & \overset{-5}{\cancel{3}} & \overset{5}{\cancel{3}} & 0 \\ 0 & 0 & -4 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 6 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & \overset{3}{\cancel{-4}} & \overset{3}{\cancel{-4}} \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 6 & 0 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & \overset{-6}{\cancel{-4}} & \overset{-6}{\cancel{-4}} \end{pmatrix}$$

* Close?!?